Two Dimensional WENO Advection Scheme

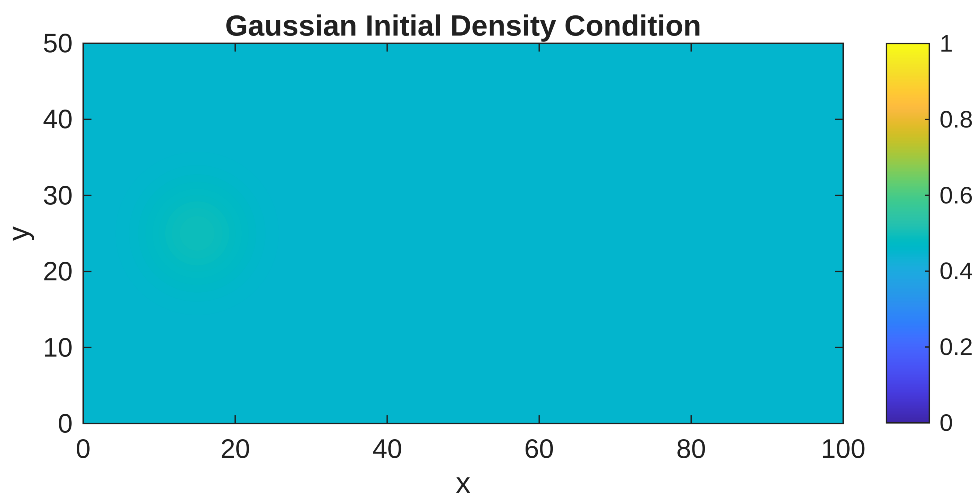
Now After successfully implementing the WENO fast sweep algorithm, i am now going to implement this 5th order WENo advection equation solver. So i will be takign two problems, and one of which is smmoth to test the workign of algorithm, and once thta is valided, i will be a takign a sharp disocontinutiy problme to test stable the algorithm is. Given below is the description of the first problem.

Problem Description

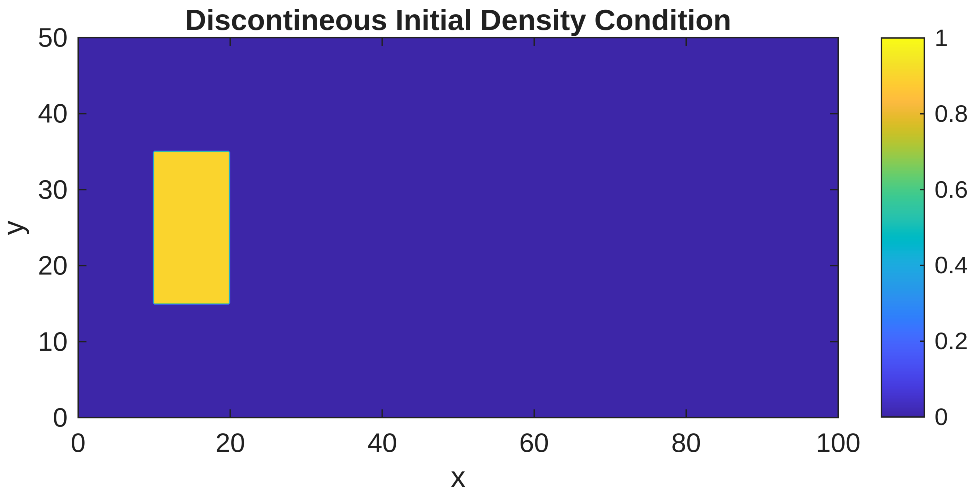
So the first problem is as follows. I am assumign a constant velocity in x dircetion and am interetsed in the numerical diffusion of the solution as it advects.

```
%-------Domain Parameter-----%
% Grid Points
global Nx;
global Ny;
Nx = 400;
Ny = 200;


% Dimension of Domain
Lx = 100;
Ly = 50;

% Domain
x = linspace(0,Lx,Nx);
y = linspace(0,Ly,Ny);
[X,Y] = meshgrid(x,y);

% Defining velocity
global x_velocity y_velocity;
x_velocity = 1*ones(Ny,Nx);
y_velocity = 0*ones(Ny,Nx);

% Space Interval
global h;
h = Lx/Nx;

% Total Time
t_total = 10;
% Time Interval
dt = 0.015;

%-------Initial Condition------%

% First Initial Condition (Gaussian Blob)

xc = 15;     % center x
yc = 25;     % center y
sigma = 5;   % width
```

```
rho_gaussian = 0.05*exp( -((X-xc).^2 + (Y-yc).^2) / (2*sigma^2) ) + 0.45;

% Second Intial Condition (Discontineous)
rho_square = zeros(size(X));
rho_square( X>=10 & X<=20 & Y>=15 & Y<=35 ) = 0.9;

% Plotting the Initial Condition
figure(1)
contourf(X,Y,rho_gaussian,'LineStyle', 'none');
title('Gaussian Initial Density Condition');
clim([0,1]);
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');
```




```
figure(2)
contourf(X,Y,rho_square,'LineStyle', 'none');
title('Discontineous Initial Density Condition');
clim([0,1]);
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');
```

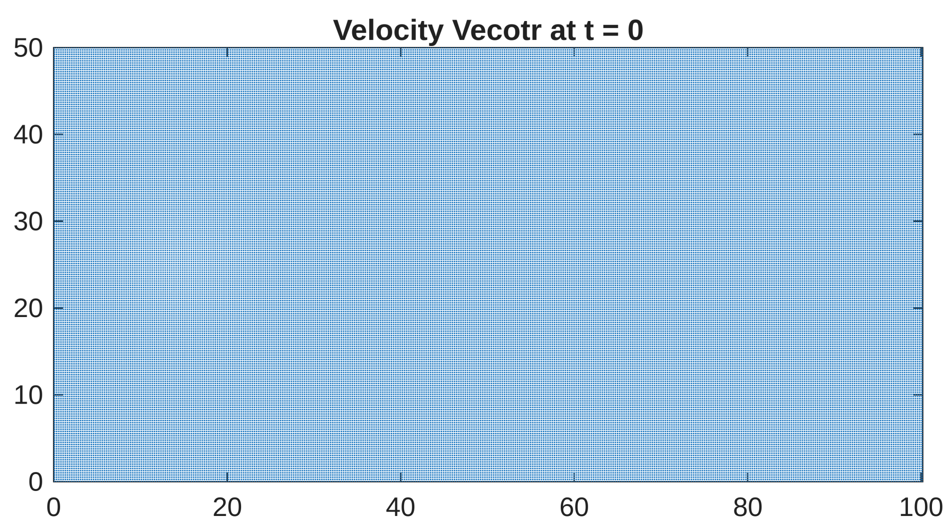


Now we will calculate the flux at each point for which i will have to assign the veloccity at each. point. So i say that everywhere the velocity of fluid is constant.A nd it is equal to **u = 1** and **v = 0**. Or the fluid is just flowing in x directuion everywhere, and there is no y component to the flow anywhere.

Since I will be trying to solve the problem where the x velocity will depend on the inverse of debsity, which means that x velocity component will be equal to (1-rho) at all location and the y velocity will be set to zero.

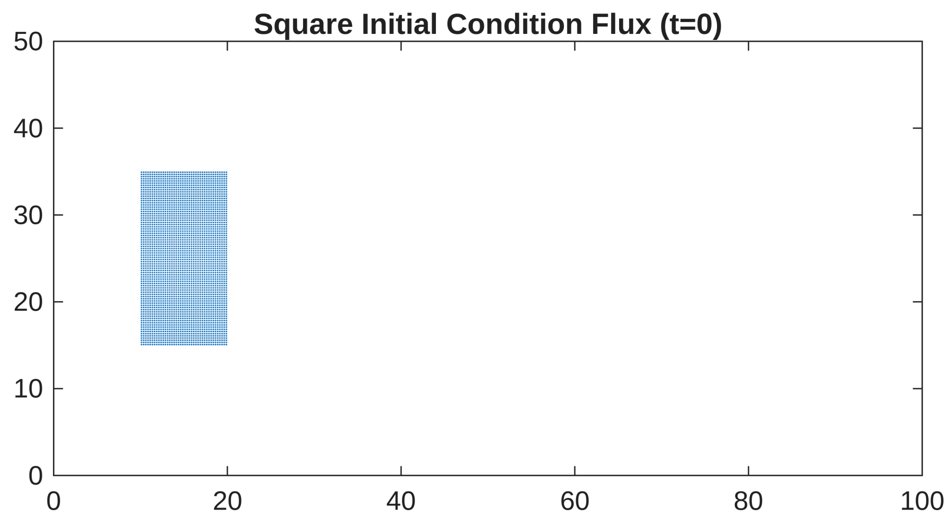
```
% Function to update X velocity
function x_velocity = makeXvel(rho)
    global x_velocity Nx Ny;
    for i = 1:1:Nx
        for j = 1:1:Ny
            x_velocity(j,i) = 1 - rho(j,i);
        end
    end
end
% Function to update Y velocity
function y_velocity = makeYvel(rho)
    global y_velocity Nx Ny;
    for i = 1:1:Nx
        for j = 1:1:Ny
            y_velocity(j,i) = 0;
        end
    end
end


function flux = makeFlux(rho)
    global Nx Ny;
    global x_velocity
```

```
    global y_velocity;

    % Initialzing Flux
    flux_x = zeros(Ny,Nx);
    flux_y = zeros(Ny,Nx);
    flux = zeros(2, Ny, Nx);

    for i = 1:1:Nx
        for j = 1:1:Ny
            flux_x(j,i) = rho(j,i) * x_velocity(j,i);
            flux_y(j,i) = rho(j,i) * y_velocity(j,i);

            flux(1,j,i) = flux_x(j,i);
            flux(2,j,i) = flux_y(j,i);
        end
    end
end

x_velocity = makeXvel(rho_gaussian);
y_velocity = makeYvel(rho_gaussian);

% --- Plotting Velocity at t=0 --- %
figure(7);
quiver(X, Y, x_velocity, y_velocity,0.5);
axis equal tight;
title('Velocity Vecotr at t = 0');
```

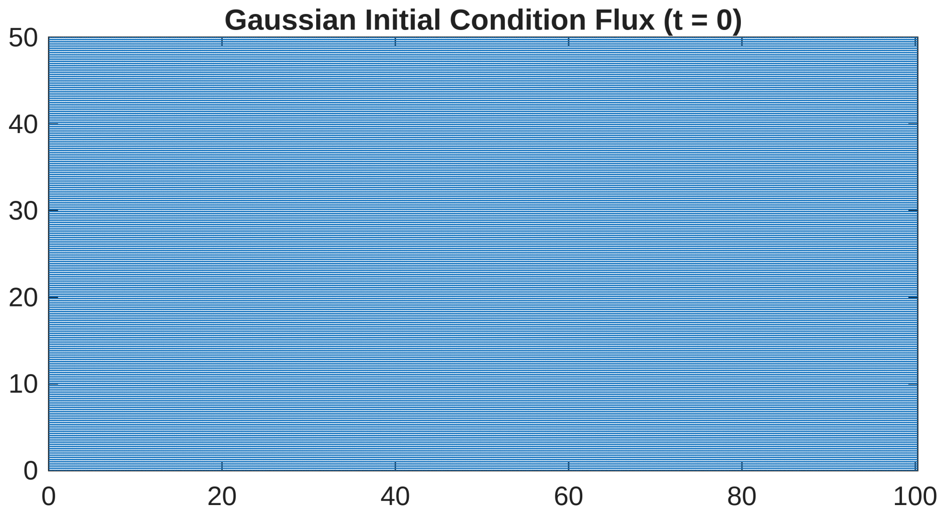



```
flux_square = makeFlux(rho_square);
flux_gaussian = makeFlux(rho_gaussian);
```

```
fx_square = squeeze(flux_square(1,:,:));
fy_square = squeeze(flux_square(2,:,:));

% --- Plotting Flux at t=0 --- %

% Plotting Discontineous Flux

figure(4);
quiver(X, Y, fx_square, fy_square,0.5);
axis equal tight;
title('Square Initial Condition Flux (t=0)');
```




```
% Plotting Gaussian Flux
fx_gauss = squeeze(flux_gaussian(1,:,:));
fy_gauss = squeeze(flux_gaussian(2,:,:));

figure(5);
quiver(X, Y, fx_gauss, fy_gauss,0.8);
axis equal tight;
title('Gaussian Initial Condition Flux (t = 0)');
```



It might not be clear,b ut these are arroaws, And they are scaled.

Now that the Flux is ready, and unstable, I will be making the function for WENO Adection equation.

One confusion i got was that if i will beb usign i+2 to i-2 to find the flux at i+1/2, then by replacing the i+1/2 with i-1/2, i will be using the values i-3 to i+1 to gte the values of flux at i-1/2. (Just replacing i with i-1). Now I am assuming that WENO is not symmetric, which means I must expect Upwind in WENO. I am still a bit skeptical about this, but lets see.

Time Marching

The loop given below is the time marching loop.

Because the loop is highly unsatble, i was asked to plot the one dimensional density VS X cordinate of domain (for constant y = 25) on the plot, and therefore the second plot will be showing the one dimensional plot (like a slice of plot from the main plot, but just at the center line).

```
% Set what kind of initial condition we must be looking at.
rho = rho_gaussian;
rho_max = 1;

%--Preallocating the memory
rho_n = zeros(Ny,Nx);
rho_1 = zeros(Ny,Nx);
rho_2 = zeros(Ny,Nx);
rho_next = zeros(Ny,Nx);

for t = 1:dt:t_total
    % -- Making rho_n and RHS of rho_n
    rho_n = rho;
```

```
    semi_discritze_form_rho_n = makeRHS(rho_n);

    %--Update the velocity
    x_velocity = makeXvel(rho_n);
    y_velocity = makeYvel(rho_n);


    % ---------- Makign the RHO_1 and then RHS of rho_1 ------ %
    for i = 4:1:Nx-2
        for j = 4:1:Ny-2
            rho_1(j,i) = rho_n(j,i) + dt *
semi_discritze_form_rho_n(j-3,i-3);
        end
    end
    % -- Extrapolation (My Intuition)
    rho_1(:,3) = rho_1(:,4); rho_1(:,2) = rho_1(:,3); rho_1(:,1) =
rho_1(:,2);  % Right Edge
    rho_1(3,:) = rho_1(4,:); rho_1(2,:) = rho_1(3,:); rho_1(1,:) =
rho_1(2,:);  % Bottom Edge
    rho_1(Ny-1,:) = rho_1(Ny-2,:); rho_1(Ny,:) =
rho_1(Ny-1,:);                  % Top Edge
    rho_1(:,Ny-1) = rho_1(:,Ny-2); rho_1(:,Ny) =
rho_1(:,Ny-1);                  % Left Edge
    % -- RHS of discritised form using rho_1
    semi_discritze_form_rho_1 = makeRHS(rho_1);



    %--------- Making the RHO_2 and RHS of RHO_2 ----------%
    for i = 4:1:Nx-2
        for j = 4:1:Ny-2
            rho_2(j,i) = 3/4 * rho_n(j,i) + 1/4 * (rho_1(j,i) + dt *
semi_discritze_form_rho_1(j-3,i-3));
        end
    end
    % -- Extrapolation (My Intuition)
    rho_2(:,3) = rho_2(:,4); rho_2(:,2) = rho_2(:,3); rho_2(:,1) =
rho_2(:,2);  % Right Edge
    rho_2(3,:) = rho_2(4,:); rho_2(2,:) = rho_2(3,:); rho_2(1,:) =
rho_2(2,:);  % Bottom Edge
    rho_2(Ny-1,:) = rho_2(Ny-2,:); rho_2(Ny,:) =
rho_2(Ny-1,:);                  % Top Edge
    rho_2(:,Ny-1) = rho_2(:,Ny-2); rho_2(:,Ny) =
rho_2(:,Ny-1);                  % Left Edge
    % -- RHS of discritised form using rho_2
    semi_discritze_form_rho_2 = makeRHS(rho_2);


    %---------Making the rho_next------------------------%
```

```
    for i = 4:1:Nx-2
        for j = 4:1:Ny-2
            rho_next(j,i) = 1/3 * rho_n(j,i) + 2/3 * (rho_2(j,i) + dt *
semi_discritze_form_rho_2(j-3,i-3));
        end
    end
    % -- Extrapolation (My Intuition)
    rho_next(:,3) = rho_next(:,4); rho_next(:,2) = rho_next(:,3);
rho_next(:,1) = rho_next(:,2);    % Right Edge
    rho_next(3,:) = rho_next(4,:); rho_next(2,:) = rho_next(3,:);
rho_next(1,:) = rho_next(2,:);    % Bottom Edge
    rho_next(Ny-1,:) = rho_next(Ny-2,:); rho_next(Ny,:) =
rho_next(Ny-1,:);                         % Top Edge
    rho_next(:,Ny-1) = rho_next(:,Ny-2); rho_next(:,Ny) =
rho_next(:,Ny-1);                         % Left Edge


    % Slicing at the 100th index
    j = 100;
    rho_slice = rho_next(j, :);

    if mod(round(t/dt),10) == 0

    % ---- 2D contour plot ---- %
        figure(6);
        contourf(x, y, rho_next, 20, 'LineColor', 'none');
        colorbar;
        clim([0 rho_max]);
        title(sprintf('Density \\rho at t = %.2f', t));
        xlabel('x'); ylabel('y');
        axis equal tight;

        % ---- 1D slice plot at y-index = 100 ---- %
        figure(8);
        plot(x, rho_slice, '*');
        grid on;
        xlabel('x');
        ylabel('\rho');
        title(sprintf('Density at Ly = 25 (Ny = 100), t = %.2f', t));
        ylim([0 rho_max]);

        drawnow;
    end


    % Substituting back the rho value \
    rho = rho_next;
end
```

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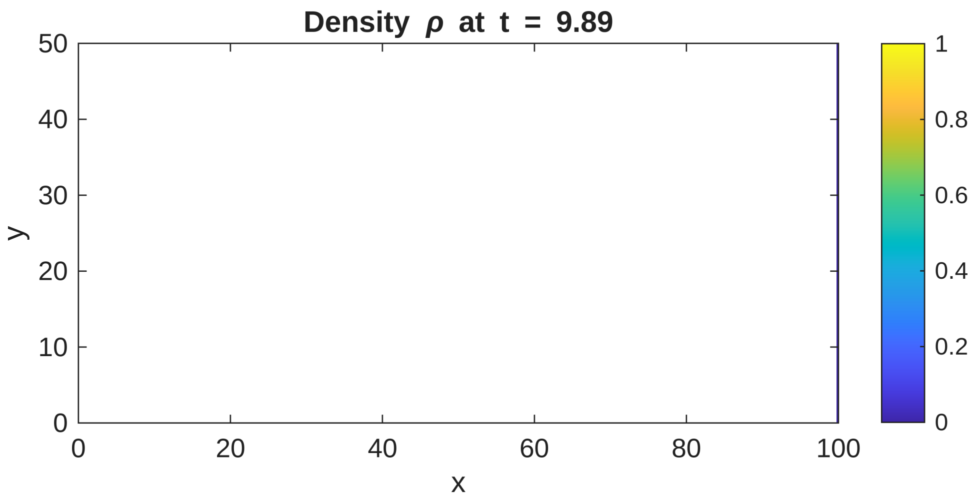

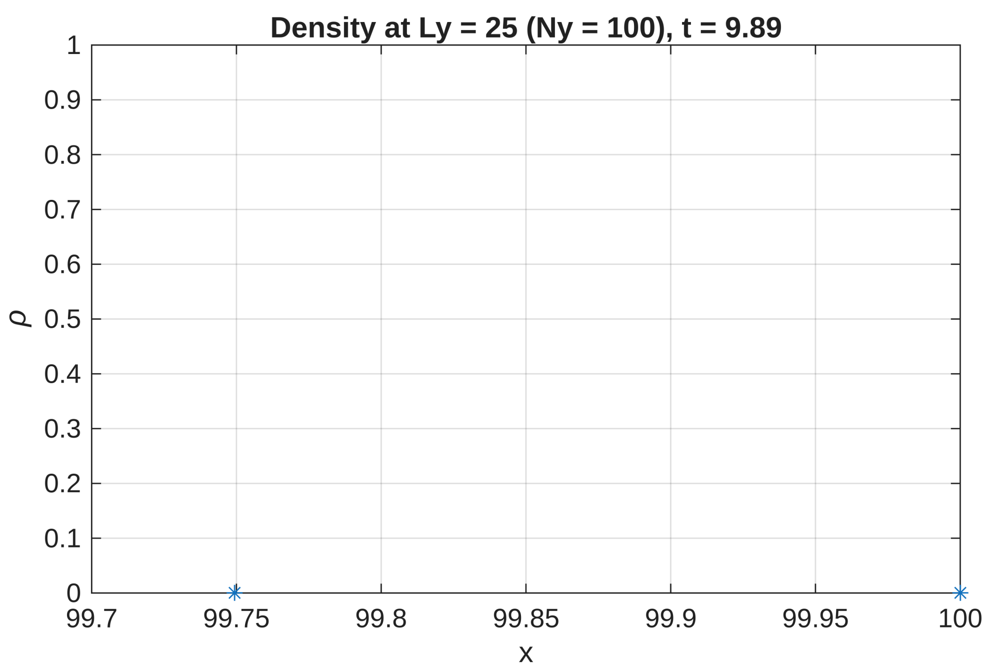

RHS of Semi discrite form

As in the paper, it uses the TVD Runga kutta time descritization. So here I will be desginign the gfunction for caluclating the L(rhgo). THis functiuon will take the value rho and then calcuate the RHS of the semi discrtise form of the adection equation fot hat rho.

```
function semi_discrite_form =  makeRHS(rho)
```

```
global Nx Ny x_velocity y_velocity h;
semi_discrite_form = zeros(Ny-5,Nx-5);

%--Flux for the given rho distribution
flux = makeFlux(rho);
flux_x = squeeze(flux(1,:,:));
flux_y = squeeze(flux(2,:,:));

% Calculating the maximum speed
alpha_x = max(abs(x_velocity(:)));
alpha_y = max(abs(y_velocity(:)));


% -- Calculating the value of f+ and f- at {i+2,i+1,i,i-1,i-2,i-3} -- %
fx_negative_whole = zeros(Ny,Nx);
fx_positive_whole = zeros(Ny,Nx);
fy_negative_whole = zeros(Ny,Nx);
fy_positive_whole = zeros(Ny,Nx);

fx_negative_half = zeros(Ny-5,Nx-5);
fx_positive_half = zeros(Ny-5,Nx-5);
fy_negative_half = zeros(Ny-5,Nx-5);
fy_positive_half = zeros(Ny-5,Nx-5);

for j = 1:1:Ny
    for i=1:1:Nx
        fx_positive_whole(j,i) = 1/2 * (flux_x(j,i) + alpha_x*rho(j,i));
        fx_negative_whole(j,i) = 1/2 * (flux_x(j,i) - alpha_x*rho(j,i));
        fy_positive_whole(j,i) = 1/2 * (flux_y(j,i) + alpha_y*rho(j,i));
        fy_negative_whole(j,i) = 1/2 * (flux_y(j,i) - alpha_y*rho(j,i));
    end
end

% -- Computing the RHS of Semi Discrite form.
for i = 4:1:Nx-2
    for j = 4:1:Ny-2

        %--X--%

        % fx{+}_[i+1/2,j]
        fx_plus_x_half_forward = stencil( ...
            fx_positive_whole(j,i-2), ...
            fx_positive_whole(j,i-1), ...
            fx_positive_whole(j,i), ...
            fx_positive_whole(j,i+1), ...
            fx_positive_whole(j,i+2));

        % fx{+}_[i-1/2,j]
        fx_plus_x_half_backward = stencil( ...
```

```
            fx_positive_whole(j,i-3), ...
            fx_positive_whole(j,i-2), ...
            fx_positive_whole(j,i-1), ...
            fx_positive_whole(j,i), ...
            fx_positive_whole(j,i+1));

        % fx{-}_[i+1/2,j]
        fx_minus_x_half_forward = stencil( ...
            fx_negative_whole(j,i+2), ...
            fx_negative_whole(j,i+1), ...
            fx_negative_whole(j,i), ...
            fx_negative_whole(j,i-1), ...
            fx_negative_whole(j,i-2));

        % fx{-}_[i-1/2,j]
        fx_minus_x_half_backward = stencil( ...
            fx_negative_whole(j,i+1), ...
            fx_negative_whole(j,i), ...
            fx_negative_whole(j,i-1), ...
            fx_negative_whole(j,i-2), ...
            fx_negative_whole(j,i-3));

        %--Y--%

        % fy{+}_[i,j+1/2]
        fy_plus_y_half_forward = stencil( ...
            fy_positive_whole(j-2,i), ...
            fy_positive_whole(j-1,i), ...
            fy_positive_whole(j,i), ...
            fy_positive_whole(j+1,i), ...
            fy_positive_whole(j+2,i));

        % fy{+}_[i,j-1/2]
        fy_plus_y_half_backward = stencil( ...
            fy_positive_whole(j-3,i), ...
            fy_positive_whole(j-2,i), ...
            fy_positive_whole(j-1,i), ...
            fy_positive_whole(j,i), ...
            fy_positive_whole(j+1,i));

        % fy{-}_[i,j+1/2]
        fy_minus_y_half_forward = stencil( ...
            fy_negative_whole(j+2,i), ...
            fy_negative_whole(j+1,i), ...
            fy_negative_whole(j,i), ...
            fy_negative_whole(j-1,i), ...
            fy_negative_whole(j-2,i));

        % fy{-}_[i,j-1/2]
        fy_minus_y_half_backward = stencil( ...
```

```
                 fy_negative_whole(j+1,i), ...
                 fy_negative_whole(j,i), ...
                 fy_negative_whole(j-1,i), ...
                 fy_negative_whole(j-2,i), ...
                 fy_negative_whole(j-3,i));

             fx_positive_half = fx_plus_x_half_forward +
fx_minus_x_half_forward;
             fx_negative_half = fx_plus_x_half_backward +
fx_minus_x_half_backward;
             fy_positive_half = fy_plus_y_half_forward +
fy_minus_y_half_forward;
             fy_negative_half = fy_plus_y_half_backward +
fy_minus_y_half_backward;

             % Computing RHS of Semi Discritized form
             semi_discrite_form(j-3,i-3) = - 1/h * (fx_positive_half -
fx_negative_half) - 1/h * (fy_positive_half - fy_negative_half);
         end
     end
end
```

This is the function used for stencil. So this will be used in calculating the value of

```
% To compute f half. Order of input matters
function f = stencil(f1,f2,f3,f4,f5)

    eta = 10^-6;

    % Compute S values
    s_1 = ((1/3) * f1) + ((-7/6) * f2) + ((11/6) * f3);
    s_2 = ((-1/6) * f2) + ((5/6) * f3) + ((1/3) * f4);
    s_3 = ((1/3) * f3) + ((5/6) * f4) + ((-1/6) * f5);

    % Compute Beta
    beta_1 = 13/12 * (f1 - 2*f2 + f3)^2 + 1/4 * (f1 - 4*f2 + 3*f3)^2;
    beta_2 = 13/12 * (f2 - 2*f3 + f4)^2 + 1/4 * (f2 - f4)^2;
    beta_3 = 13/12 * (f3 - 2*f4 + f5)^2 + 1/4 * (3*f3 - 4*f4 + f5)^2;

    % Definign Gamma
    gamma_1 = 1/10;
    gamma_2 = 3/5;
    gamma_3 = 3/10;

    % Computing weights bar
    w_1_dash = ( gamma_1 ) / ((eta + beta_1)^2);
    w_2_dash = ( gamma_2 ) / ((eta + beta_2)^2);
    w_3_dash = ( gamma_3 ) / ((eta + beta_3)^2);

    % Computing Weights
```

```
    w_1 = ( w_1_dash ) / (w_1_dash + w_2_dash + w_3_dash);
    w_2 = ( w_2_dash ) / (w_1_dash + w_2_dash + w_3_dash);
    w_3 = ( w_3_dash ) / (w_1_dash + w_2_dash + w_3_dash);

    % Computing Flux
    f = w_1 * s_1 + w_2 * s_2 + w_3 * s_3;
end
```